1. A good approximation to the metric outside the surface of the Earth is provided by

$$\mathrm{d}s^2 = -(1+2\Phi)\mathrm{d}t^2 + (1-2\Phi)\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2)\,, \tag{1}$$

where the gravitational potential,

$$\Phi = -\frac{GM}{r}\,, \tag{2}$$

(familiar from Newtonian gravity) may be assumed to be small. Here G is Newton's constant and M is the mass of the Earth.

   (a) Imagine there is a clock on the surface of the Earth at distance R_1 from the Earth's centre and another clock on a tall building at distance R_2 from the Earth's centre. Calculate the time elapsed on each clock as a function of the coordinate time t. Which clock moves faster?

   (b) Solve for a geodesic corresponding to a circular orbit, around the equator of the Earth ($\theta = \pi/2$). What is $d\phi/dt$?

   (c) How much proper time elapses while a satellite at radius R_1 (skimming along the surface of the Earth, neglecting air resistance etc.) completes one orbit? (You can work to first order in Φ) Plug in the actual numbers for the radius of the Earth and so on to get an answer in seconds. How does this number compare to the proper time elapsed on the clock stationary on the surface?

1. We observe the universe, when averaged on large-enough scales, to be approximately homogeneous (it is the same everywhere) and isotropic (it looks the same in every direction: these are two different things!). It turns out that if you also require that space be flat (not spacetime), the most general metric which obeys these symmetries is the so-called Friedmann-Lemaître-Robertson-Walker (FLRW) metric with the line element

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\left(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\right) ,$$

where $a(t)$ is called the scale factor. It describes how distances scale as a function of time.

It is also useful to define the Hubble parameter

$$H \equiv \frac{\dot{a}}{a} ,$$

which describes the rate of change of the scale of the universe. $H > 0$ means that the universe is expanding. Imagine the universe is filled with a perfect fluid of energy density ρ and pressure p. Another useful quantity to define is the equation-of-state parameter

$$w \equiv \frac{p}{\rho} ,$$

which frequently is taken to be a constant. Take the fluid to be at rest in the coordinate frame.

(a) By working out the covariant conservation equation for the energy-momentum tensor, show that the equation for the evolution of the energy density is

$$\dot{\rho} + 3H(\rho + p) = 0$$

For what value of w is the energy density constant in an expanding universe? This is the equation of state of the cosmological constant.

(b) Assuming that $w = \text{const}$, solve the energy density conservation equation to show that $\rho \propto a^{-3(1+w)}$.

(c) Calculate the Einstein equations for this system (there are two distinct ones). Solve for the evolution of the scale factor as a function of time for a fluid with a constant w. These two equations are called the Friedmann equations.

(d) Accelerated expansion of the universe as $\ddot{a}/a > 0$. What is the condition on the equation of state which allows acceleration to occur?